An Analytical Approach to One-dimensional Solute Dispersion along and against Transient Groundwater flow in Aquifers

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Abstract
A one-dimensional advective-dispersive equation is solved analytically to predict patterns of contaminant concentration distribution in a finite or semi-infinite homogeneous aquifer. The dispersion of solute along and against transient groundwater flow is considered. Initially, the aquifer is assumed to be not clean, which means that some initial background concentration exists and it is represented by a uniform or exponentially decreasing form of concentration. A pulse-type exponentially decreasing temporally dependent source concentration is considered in the intermediate portion of the aquifer and at other end, the concentration gradient is supposed to be zero. The Laplace Transform Technique (LTT) is used to obtain the solution of the problem of contaminant distribution in two different domains. In one domain the distribution pattern is depicted along groundwater flow and in other domain, it is depicted against the groundwater flow. This represents a realistic situation of the contaminant concentration distribution pattern in an aquifer in the presence or absence of temporally dependent source concentration. The time varying velocity expressions are considered in which a sinusoidal form of velocity occurs often in tropical regions of India. The dispersion is directly proportional to the seepage velocity used in which the effect of molecular diffusion is not taken into account because the value of molecular diffusion does not vary significantly for different types of soil and contaminant behavior. Results of the obtained analytical solution may form useful complements to benchmark numerical models or for their verification. The long term vertical transport of solute during transient saturated groundwater flow can be described correctly with appropriate analytical models. The analytical solution may also be used as a preliminary predictive tool for groundwater resource management to estimate transport parameters.

Keywords: Contaminant, Homogeneous, Semi-infinite, porous media, Analytical Solution.

1. Introduction
The water demand in the world is rapidly increasing due to population growth, extensive industrialization, rising standard of living, growing energy generation, and intensifying agriculture. There are two major sources to fulfill these demands, surface water and groundwater. Out of these two, groundwater plays an important role in the meeting this ever increasing
demand. It also plays a vital role in the economy of many developing countries such as India, and in ensuring their food and energy security. Surface and groundwater pollution in mining and industrial areas is a serious concern. Jharia coalfield is one of the biggest coalfields and highly industrialized areas in India. In such areas, water gets contaminated by artificial recharge, waste disposal in wet excavation, mine water, industrial effluents, tailing ponds etc. (Abhishek et al., 2006). The natural and generally high quality groundwater is under attack today by many sources and types of contaminants which are associated with human activities and land use (Gelhar & Wilson, 1974). Therefore, groundwater systems, planning, and management are needed for judicious use of groundwater. During last four decades, solute transport problems have attracted considerable attention from geologists, environmentalists, civil engineers, and scientists, etc. Solute transport problems have been solved using various methods.

Mathematical models play an important role in the analysis of the transport of contaminants in groundwater. These models are generally based on the governing equation of flow under variably saturated conditions and the classic advective-dispersive equation. Groundwater transport and its mathematical models were presented by Fried (1975) and Javendal and Tsang (1984). The theory and practice of deterministic modeling of groundwater processes were provided by Anderson and Woessner (1992) and; Bear and Verrujit (1987). One of the difficulties of predicting the movement of contaminants lies in our lack of ability to solve these equations accurately and efficiently for general cases (Yeh et al., 1993). Although many transport problems have been solved numerically, analytical solutions are still pursued by many scientists because they provide better physical insights. Zheng and Bennett (1995) noted that analytical solutions are usually derived from basic physical principles and are free from numerical dispersion and other truncation errors that often occur in numerical simulation. Chrysikopoulos and Sim (1996) presented a one-dimensional virus transport in homogeneous porous media with time-dependent distribution coefficient. Tartakovsky and Di Federico (1997) also obtained an analytical solution for contaminant transport in non-uniform flow. An analytical solution for groundwater transit time through unconfined aquifers was presented by Chesnaux et al. (2005). Singh et al. (2008) presented an analytical solution in homogeneous semi-infinite aquifer with Dirichlet boundary condition. A solute transport for one-dimensional homogeneous porous formations with time dependent point-source concentration was presented by Singh et al. (2009). Singh et al. (2010) also present an analytical solution for solute transport along and against time dependent source concentration in homogeneous finite aquifers.

The objective of the present work is to obtain an analytical solution of one-dimensional solute transport with transient groundwater flow. The aquifer is considered semi-infinite and homogeneous. A temporally dependent pulse type input concentration is considered at the intermediate portion of the aquifer. Since groundwater moves very slowly, we consider two cases: (1) the distribution of solute along as well as (2) against the groundwater flow. The analytical solution is obtained using the Laplace Transform Technique (LTT). Time varying velocity expressions are considered for a numerical example and discussions.

2. Mathematical Formulation

Let \( c(x,t) [\text{ML}^{-3}] \) be the solute concentration at position \( x [\text{L}] \) at time \( t [\text{T}] \) in a homogeneous semi-finite aquifer. Let \( u [\text{LT}^{-1}] \) and \( D [\text{L}^2 \text{T}^{-1}] \) be the groundwater velocity and dispersion coefficient at any time \( t \). The solute transport through a saturated porous media can be described by a partial differential equation (PDE) of parabolic type. This equation is usually known as advective-dispersive equation. In one-dimensional space a linear advective-dispersive equation can be written as (Bear, 1972):
where \( u_0 [LT^{-1}] \) is the initial groundwater velocity. Here \( V(t) \) is considered as a non-dimensional expression such as (i) \( V(t) = 1 - \sin mt \) and (ii) \( V(t) = \exp(-mt) \), \( mT^{-1} \) is the flow resistance coefficient. Initially, the aquifer is considered not solute free, i.e., aquifer is not clean, so, the initial background contaminant concentration must exist in the aquifer which is represented by a uniform concentration \( c_i [ML^{-3}] \) at time \( t = 0 \). The temporally dependent input source concentration in the form of combination of constant and exponentially decreasing function, i.e., \( c_0 [1+\exp(-qt)] \) is considered in the intermediate portion of the aquifer, i.e., at \( x = x_0 \) as shown in fig. (1) till \( t = t_0 \) and beyond that it becomes zero. Here \( c_0 [ML^{-3}] \) is the solute concentration and \( q [T^{-1}] \) is a decay parameter. For this reason, two cases arise: i) the concentration distribution behavior of the solute in the region \( x \geq x_0 \), i.e., solute dispersion along groundwater flow and ii) concentration distribution of the solute in the region \( 0 \leq x \leq x_0 \), i.e., solute dispersion against groundwater flow. Let the solute concentration gradient at the other end of the aquifer is supposed to be zero.

Fig.1. Physical Schematic of the problem

2.1 Analytical solution for concentration distribution behavior of solute in the region \( x \geq x_0 \)

The initial and boundary condition of the mathematical model for the problem can be written as follows:

\[
c(x,t) = c_i; \; x \geq x_0, \; t = 0
\]
where \( u_0 \) is the initial groundwater velocity at each \( x \). The dispersion coefficient varies approximately directly with the seepage velocity for various types of porous media (Ebach & White, 1958). Also, it was found that such a relationship established for steady flow was valid for unsteady flow with the sinusoidal varying seepage velocity (Rumer, 1962). Let \( D = au \) where the coefficient of dimension length is \( a \), and it depends upon the pore system geometry and average pore size diameter of the porous medium. Using equation (2), we get \( D_0 = au_0 \) where \( D_0 \) is an initial dispersion coefficient. Put \( X = x - x_0 \) then PDE (1) and equations (3)\(-\)(4) can be written as

\[
D_0 \frac{\partial^2 c}{\partial X^2} - u_0 \frac{\partial c}{\partial X} = \frac{1}{V(t)} \frac{\partial c}{\partial t} \tag{5}
\]

\[
c(X, t) = c_0; X \geq 0, t = 0 \tag{6}
\]

\[
c(X, t) = \left\{ \begin{array}{ll}
c_0 (2 - qt); & 0 < t \leq t_0 \\
0; & t > t_0
\end{array} \right.; X = 0 \tag{7}
\]

\[
\frac{\partial c}{\partial X} = 0; t \geq 0, X \to \infty \tag{8}
\]

Introducing a new time variable \( T^* \) by the transformation (Crank, 1975) as

\[
T^* = \int_0^t V(t) dt \tag{9}
\]

Equation (5) becomes

\[
D_0 \frac{\partial^2 c}{\partial X^2} - u_0 \frac{\partial c}{\partial X} = \frac{\partial c}{\partial T^*} \tag{10}
\]

Now the set of non-dimensional variables can be written as

\[
Y = \frac{X u_0}{D_0}, C = \frac{c}{c_0}, T = \frac{u_0^2 T^*}{D_0}, Q = \frac{q D_0}{u_0^2} \tag{11}
\]

The PDE (10) and corresponding initial and boundary conditions in non-dimensional form can now be written as follows:

\[
\frac{\partial^2 C}{\partial Y^2} = \frac{\partial C}{\partial T} \tag{12}
\]

\[
C(Y, T) = \frac{c_0}{c_0}; Y \geq 0, T = 0 \tag{13}
\]

\[
C(Y, T) = \left\{ \begin{array}{ll}
(2 - QT); & 0 < T \leq T_0 \\
0; & T > T_0
\end{array} \right.; Y = 0 \tag{14}
\]

\[
\frac{\partial C}{\partial Y} = 0; T \geq 0, Y \to \infty \tag{15}
\]

Using the transformation \( C(Y, T) = K(Y, T) \exp \left( \frac{Y - T}{2} \right) \) in equations (12)\(-\)(15) and applying the Laplace transform, we can get the solution of the obtained boundary value problem as follows:
\[ \overline{K}(Y, p) = \left[ \frac{2}{(p-\frac{1}{4})} - \frac{Q}{(p-\frac{1}{4})^2} \right] \left( 1 - \exp\left[ -\left( p - \frac{1}{4} \right) T_0 \right] \right) + \frac{Q T_0}{(p-\frac{1}{4})} \exp\left[ -\left( p - \frac{1}{4} \right) T_0 \right] - \frac{c_i}{c_0} \frac{1}{(p-\frac{1}{4})} \exp(-Y \sqrt{p}) \]

\[ + \frac{c_i}{c_0} \exp\left( -\frac{Y}{2} \right) \]

\[ \overline{K}(Y, p) = \left[ 2 \overline{K}_1(Y, p) - Q \overline{K}_2(Y, p) + QT_0 \overline{K}_3(Y, p) - \frac{c_i}{c_0} \overline{K}_4(Y, p) + \frac{c_i}{c_0} \frac{\exp(-Y/2)}{(p-\frac{1}{4})} \right] \tag{17} \]

where

\[ \overline{K}_1(Y, p) = \left( \frac{1}{p-\frac{1}{4}} \right) \left[ 1 - \exp\left( -\left( p - \frac{1}{4} \right) T_0 \right) \right] \exp(-Y \sqrt{p}) \tag{19a} \]

\[ \overline{K}_2(Y, p) = \left( \frac{1}{p-\frac{1}{4}} \right)^2 \left[ 1 - \exp\left( -\left( p - \frac{1}{4} \right) T_0 \right) \right] \exp(-Y \sqrt{p}) \tag{19b} \]

\[ \overline{K}_3(Y, p) = \left( \frac{1}{p-\frac{1}{4}} \right) \exp\left( -\left( p - \frac{1}{4} \right) T_0 \right) \exp(-Y \sqrt{p}) \tag{19c} \]

\[ \overline{K}_4(Y, p) = \frac{\exp(-Y \sqrt{p})}{(p-\frac{1}{4})} \tag{19d} \]

Taking the inverse Laplace transform of equation (18), we get

\[ K(Y, T) = \left[ 2K_1(Y, T) - QK_2(Y, T) + QT_0K_3(Y, T) - \frac{c_i}{c_0}K_4(Y, T) + \frac{c_i}{c_0} \exp\left( \frac{T}{4} - \frac{Y}{2} \right) \right] \tag{20} \]

\[ K_1(Y, T) = \left\{ \begin{array}{ll} F(Y, T); & 0 < T \leq T_0 \\ F(Y, T) - F(Y, T - T_0); & T > T_0 \end{array} \right\} \tag{21a} \]

\[ K_2(Y, T) = \left\{ \begin{array}{ll} G(Y, T); & 0 < T \leq T_0 \\ G(Y, T) - G(Y, T - T_0); & T > T_0 \end{array} \right\} \tag{21b} \]

\[ K_3(Y, T) = \left\{ \begin{array}{ll} 0; & 0 < T \leq T_0 \\ [F(Y, T) - F(Y, T - T_0)]; & T > T_0 \end{array} \right\} \tag{21c} \]

\[ K_4(Y, T) = F(Y, T) \tag{21d} \]

Therefore the solution is obtained as follows:
\[ C(Y, T) = \exp \left( \frac{Y}{2} - \frac{T}{4} \right) \left[ \left( 2 + \frac{c_i}{c_0} \right) F(Y, T) - \frac{c_i}{c_0} \exp \left( \frac{T}{4} - \frac{Y}{2} \right); 0 < T \leq T_0 \right. \]
\[ \left. \frac{c_i}{c_0} \exp \left( \frac{T}{4} - \frac{Y}{2} \right); T > T_0 \right] \]

where
\[ F(Y, T) = \frac{1}{2} \left[ \exp \left( \frac{T}{4} - \frac{Y}{2} \right) \text{erfc} \left( \frac{Y + \sqrt{T}}{2} \right) + \exp \left( \frac{T}{4} + \frac{Y}{2} \right) \text{erfc} \left( \frac{Y - \sqrt{T}}{2} \right) \right] \]
\[ G(Y, T) = \frac{1}{2} \left[ (T - Y) \exp \left( \frac{T}{4} - \frac{Y}{2} \right) \text{erfc} \left( \frac{Y + \sqrt{T}}{2} \right) + (T + Y) \exp \left( \frac{T}{4} + \frac{Y}{2} \right) \text{erfc} \left( \frac{Y - \sqrt{T}}{2} \right) \right] \]

One may express the obtained solution (22) in terms of the original space and time variable, i.e., \( x \) and \( t \) by using equation (11) and the transformation \( x = X + x_0 \).

### 2.2 Analytical solution for concentration distribution behavior of solute in the region \( 0 \leq x \leq x_0 \)

In this domain, the initial and boundary conditions of the problem can now be written as
\[ c(x, t) = c_i; \quad x > 0, t = 0 \] \[ \frac{\partial c}{\partial x} = 0; \quad x = 0, t \geq 0 \] \[ c(x, t) = c_0 \left[ 1 + \exp(-qt) \right]; 0 < t \leq t_0 \] \[ x = x_0; t > t_0 \] \[ \frac{\partial c}{\partial x} = 0; x = 0, t \geq 0 \]

Using equation (2) and the relation between seepage velocity and dispersion i.e., \( D = au \) then the PDE given in equation (1) can be written as
\[ \frac{D_0}{\partial x^2} - u_0 \frac{\partial c}{\partial x} = \frac{1}{V(t)} \frac{\partial c}{\partial t} \]

The new time variable given in equation (9) transforms equation (26) as
\[ \frac{D_0}{\partial x^2} - u_0 \frac{\partial c}{\partial x} = \frac{\partial c}{\partial T^*} \]

Now the set of non-dimensional variables are defined as follows:
\[ X = \frac{x}{x_0}, C = \frac{c}{c_0}, T = \frac{D_0}{x_0^2} T^*, U = \frac{u_0 x_0}{D_0}, Q = \frac{q x_0^3}{D_0} \]

Equation (27) and the corresponding initial and boundary conditions given in equations (24) to (25) in non-dimensional form can be written as
\[ \frac{\partial^2 C}{\partial X^2} - U \frac{\partial C}{\partial X} = \frac{\partial C}{\partial T} \]
\[ C(X, T) = \frac{c_i}{c_0}; X > 0, T = 0 \]
\[ \frac{\partial C}{\partial X} = 0; X = 0, T \geq 0 \]
\[ C(X, T) = \begin{cases} 2^{-QT}; & 0 < T \leq T_0 \\ 0; & T > T_0 \end{cases} \quad ; X = 1 \] (32)

Using the transformation
\[ C(X, T) = K(X, T) \exp \left( \frac{UX}{2} - \frac{U^2 T}{4} \right) \] (33)

in equations (29) to (32) and applying the Laplace transform and its inverse, we can get the solution of the boundary value problem as follows:

\[
K(X, T) = \exp \left( -\frac{U}{2} \right) \left[ 2 \left( K_1(X, T) + K_2(X, T) - K_3(X, T) \right) - Q \left( K_4(X, T) + K_5(X, T) - K_6(X, T) \right) \right] \\
+ Q T_0 \left( K_7(X, T) + K_8(X, T) - K_9(X, T) \right) - \frac{c}{c_0} \left( K_{10}(X, T) + K_{11}(X, T) - K_{12}(X, T) \right) \\
\frac{c}{c_0} \exp \left( \frac{U^2 T}{4} - \frac{UX}{2} \right) 
\]

where

\[ K_1(X, T) = \begin{cases} F(1-X, T) + F(1-X, T-T_0); & 0 < T \leq T_0 \\ F(1-X, T) - F(1-X, T-T_0); & T > T_0 \end{cases} \] (37a)

\[ K_2(X, T) = \begin{cases} I(1-X, T-T_0); & 0 < T \leq T_0 \\ I(1-X, T-T_0); & T > T_0 \end{cases} \] (37b)

\[ K_3(X, T) = \begin{cases} F(1+X, T-T_0); & 0 < T \leq T_0 \\ I(1+X, T-T_0); & T > T_0 \end{cases} \] (37c)

\[ K_4(X, T) = \begin{cases} F(1+X, T-T_0); & 0 < T \leq T_0 \\ I(1+X, T-T_0); & T > T_0 \end{cases} \] (37d)

\[ K_5(X, T) = \begin{cases} F(1+X, T-T_0); & 0 < T \leq T_0 \\ I(1+X, T-T_0); & T > T_0 \end{cases} \] (37e)

\[ K_6(X, T) = \begin{cases} F(1+X, T-T_0); & 0 < T \leq T_0 \\ I(1+X, T-T_0); & T > T_0 \end{cases} \] (37f)

\[ K_7(X, T) = \begin{cases} F(1+X, T-T_0); & 0 < T \leq T_0 \\ I(1+X, T-T_0); & T > T_0 \end{cases} \] (37g)

\[ K_8(X, T) = \begin{cases} F(1+X, T-T_0); & 0 < T \leq T_0 \\ I(1+X, T-T_0); & T > T_0 \end{cases} \] (37h)

\[ K_9(X, T) = \begin{cases} F(1+X, T-T_0); & 0 < T \leq T_0 \\ I(1+X, T-T_0); & T > T_0 \end{cases} \] (37i)

\[ K_{10}(X, T) = F(1-X, T) \] (37j)
\[ K_{11}(X,T) = F(1+X,T) - F(3-X,T) + UG(1+X,T) - UG(3-X,T) \] (37k)

\[ K_{12}(X,T) = F(3+X,T) + 2UG(3+X,T) + U^2H(3+X,T) \] (37l)

The solution of the boundary value problem can now be written as:

\[
K(X,T) = \begin{cases}
\begin{bmatrix}
\frac{2-c_i}{c_0} \left[ F(1-X,T) + F(1+X,T) - F(3-X,T) - F(3+X,T) \right] \\
+ UG(1+X,T) - UG(3-X,T) - 2UG(3+X,T) + U^2H(3+X,T) \\
Q \left[ I(1-X,T) - I(1+X,T) - I(3-X,T) + I(3+X,T) \right]
\end{bmatrix} \\
\frac{c_i}{c_0} \exp \left( \frac{UX}{2} - \frac{U^2T}{4} \right), \quad 0 < T \leq T_0
\end{cases}
\]

\[
\begin{cases}
\begin{bmatrix}
\frac{2-c_i}{c_0} \left[ F(1-X,T) - F(1+X,T) - T_0 \right] \\
- \frac{1}{2} [F(3-X,T) - F(3-X,T - T_0)] \\
- U \left[ G(1+X,T) - G(3-X,T) - T_0 \right] \\
- U \left[ G(3-X,T) - G(3-X,T - T_0) \right] \\
- 2U \left[ I(3-X,T) - I(3-X,T - T_0) \right] \\
- 2U \left[ I(3-X,T - T_0) - I(3-X,T) \right] \\
+ \frac{c_i}{c_0} \exp \left( \frac{UX}{2} - \frac{U^2T}{4} \right), \quad T > T_0
\end{bmatrix}
\end{cases}
\]

where

\[ F(X,T) = \frac{1}{2} \left\{ \exp \left( \frac{U^2T}{4} - \frac{UX}{2} \right) \text{erfc} \left( \frac{X}{2\sqrt{T}} - \frac{U\sqrt{T}}{2} \right) + \exp \left( \frac{U^2T}{4} + \frac{UX}{2} \right) \text{erfc} \left( \frac{X}{2\sqrt{T}} + \frac{U\sqrt{T}}{2} \right) \right\} \] (39a)

\[ G(X,T) = \frac{T}{\pi} \exp \left( -\frac{X^2}{4T} \right) - \frac{1}{2U} \exp \left( \frac{U^2T}{4} + \frac{UX}{2} \right) \text{erfc} \left( \frac{X}{2\sqrt{T}} + \frac{U\sqrt{T}}{2} \right) \] (39b)

\[ H(X,T) = \frac{1}{2U^2} \exp \left( \frac{U^2T}{4} + \frac{UX}{2} \right) \text{erfc} \left( \frac{X}{2\sqrt{T}} + \frac{U\sqrt{T}}{2} \right) + \frac{U}{\sqrt{\pi}} \left( \frac{UX}{4} - \frac{U^2T}{2} \right) \exp \left( -\frac{X^2}{4T} \right) \] (39c)
One can express the solution in terms of the original space and time variable, i.e., $x$ and $t$ by using equation (28) and the transformation $x = x_0 X$.

3. Numerical Result and Discussion
   Let us consider the sinusoidal and exponential forms of velocities as follows:

$$C(X,T) = \begin{cases} 
\exp \left( \frac{(U(X-1) - U^2 T) - 4}{4} \right) \left[ e^{\frac{2}{c_0}} \left( F(1-X,T) + F(1+X,T) - F(3-X,T) - F(3+X,T) \right) \right] & \frac{c_i}{c_0}, 0<T<T_0 \\
\exp \left( \frac{(U(X-1) - U^2 T) - 4}{4} \right) \left[ e^{\frac{2}{c_0}} \left( F(1-X,T) - F(3-X,T) - F(3+X,T) - F(3-X,T) \right) \right] & \frac{c_i}{c_0}, T>T_0
\end{cases}$$

$$I(X,T) = \frac{1}{2U} \left\{ \begin{array}{l} (X+UT) \exp \left( \frac{U^2 T}{4} + \frac{UX}{2} \right) \text{erfc} \left( \frac{X}{2\sqrt{T}} + \frac{U\sqrt{T}}{2} \right) \\
-(X-UT) \exp \left( \frac{U^2 T}{4} - \frac{UX}{2} \right) \text{erfc} \left( \frac{X}{2\sqrt{T}} - \frac{U\sqrt{T}}{2} \right) \end{array} \right\}$$

$$J(X,T) = \frac{1}{2U} \sqrt{\frac{T}{\pi}} \left( \frac{1-UX + U^2 T}{2} \right) \exp \left( -\frac{X^2}{4T} \right) - \frac{1}{2U} \left( U^2 T + UX - 1 \right) \exp \left( \frac{U^2 T}{4} + \frac{UX}{2} \right) \text{erfc} \left( \frac{X}{2\sqrt{T}} + \frac{U\sqrt{T}}{2} \right)$$

$$+ \frac{1}{2U} \left( U^2 T - 1 + \frac{U^2}{2} (X-UT)^2 \right) \exp \left( \frac{U^2 T}{4} - \frac{UX}{2} \right) \text{erfc} \left( \frac{X}{2\sqrt{T}} - \frac{U\sqrt{T}}{2} \right)$$

$$L(X,T) = \frac{1}{U} \left\{ \begin{array}{l} \left[ 1-\frac{U}{2}(U^2 T - 1) (X-UT) \right] \frac{U^3}{6} (X-UT)^3 \exp \left( \frac{U^2 T}{4} + \frac{UX}{2} \right) \text{erfc} \left( \frac{X}{2\sqrt{T}} + \frac{U\sqrt{T}}{2} \right) \\
\left[ 1-\frac{U}{2} (X-UT)^2 \right] \exp \left( \frac{U^2 T}{4} + \frac{UX}{2} \right) \text{erfc} \left( \frac{X}{2\sqrt{T}} + \frac{U\sqrt{T}}{2} \right) \\
+ 2U \left[ 3+U^2 T + \frac{U^2}{4} (X-UT)^2 \right] \exp \left( \frac{X^2}{4T} \right) \end{array} \right\}$$

Finally, $C(X,T)$ can be written as follows:

One can express the solution in terms of the original space and time variable, i.e., $x$ and $t$ by using equation (28) and the transformation $x = x_0 X$.
\[
V(t) = 1 - \sin mt \quad (41a)
\]
\[
V(t) = \exp(-mt), \quad mt < 1 \quad (41b)
\]

where \( m (\text{day}^{-1}) \) is the flow resistance coefficient. For both expressions, the non-dimensional time variable \( T \) for the region \( x \geq x_0 \) may be written as

\[
T = \frac{u_0^2}{mD_0} \left[ mt - (1 - \cos mt) \right] \quad (42a)
\]
\[
T = \frac{u_0^2}{mD_0} \left[ 1 - \exp(-mt) \right] \quad (42b)
\]

The same set of expressions in non-dimensional time variable form \( T \) for the domain \( 0 \leq x \leq x_0 \) may be written as follows:

\[
T = \frac{D_0}{m^2x_0^2} \left[ mt - (1 - \cos mt) \right] \quad (43a)
\]
\[
T = \frac{D_0}{m^2x_0^2} \left[ 1 - \exp(-mt) \right] \quad (43b)
\]

Here, \( mt = 3k + 2 \) is chosen where \( k \) represents the whole number. Now for \( m = 0.0165 \ (\text{day}^{-1}) \), equation (41a) approximately yields; \( t = 182k + 121 \). For these values of \( mt \), velocity \( u \), is alternatively minimum or maximum. Hence, it represents the minimum groundwater level and velocity during the month of June and maximum during December just after six months in one year. The next data of \( t \) represents minimum and maximum records during June and December, respectively, in subsequent years. Analytical solutions given by equations (22) and (40) are computed for values \( c_i = 0.05, c_0 = 1.0, u_0 = 0.001 \ (km/days), x_0 = 30km \), with \( D_0 = 0.4 \ km^2/days \), and \( D_0 = 0.04 \ km^2/days \). The concentration values in the presence of source pollution till \( t = t_0 \ (1500 \ days) \) are depicted graphically in the presence of time dependent concentration of contaminants for \( q = 0.0001 \ (\text{day}^{-1}) \) at \( mt = 3k + 2, 2 \leq k \leq 7 \) which represents minimum and maximum values of groundwater level and velocity during June and December in 2nd, 3rd and 4th years at the respective time values \( t \). After the source is eliminated the solution is computed with \( q = 0.00001 \ (\text{day}^{-1}) \ at \ mt = 3k + 2 \), where \( 8 \leq k \leq 13 \) which represents the duration of June and December alternatively in the 5th, 6th and 7th years, respectively. The concentration distribution behaviour of contaminants along unsteady flow of the sinusoidal form of velocity is given in equation (41a) depicted in fig. 2(a) when \( 0 < t \leq t_0 \) and fig. 2(b) when \( t > t_0 \). From the present study it is observed that the contaminant concentration decreases with time and distance traveled in the presence of source contaminants. At \( x = x_0 \) the temporally dependent source concentration decreases as time increases and emerges at the common point near the source. The solute concentration goes on decreasing in the domain \( x \geq x_0 \). It is also observed that the trend is almost similar in the domain \( 0 \leq x \leq x_0 \). When \( t > t_0 \), i.e., once the source concentration is eliminated, the contaminant concentration increases initially and attains the maximum level and then starts to decrease and goes on decreasing and tends towards minimum and harmless concentration in the domains \( x \geq x_0 \) and \( 0 \leq x \leq x_0 \). For the same set of inputs, except for \( m = 0.0002 \ (\text{day}^{-1}) \), as \( mt < 1 \), equations (22) and (40) were also computed for the
exponentially decreasing form of velocity given in equation (41b). It is observed that the contaminant concentration follows almost the same trends in the presence and absence of source contaminants, respectively in both the domains. The decreasing tendency of contaminant concentration with time and distance traveled is depicted graphically in fig. 3(a) when $0 < t \leq t_0$ and fig. 3(b) when $t > t_0$ for the exponentially decreasing form of velocity. It is observed the rate of decreasing tendency for the sinusoidal form of velocity is slightly slower than the exponential form of velocity in both the domain.

Fig. 2(a). Pollutant concentration distribution along the flow with $D_0 = 0.4 \text{ km}^2/\text{day}$ and against the flow with $D_0 = 0.04 \text{ km}^2/\text{day}$ for unsteady groundwater flow with the sinusoidal form of velocity at $q=0.0001/\text{day}$ for $0 < t \leq t_0$.

Fig. 2(b). Pollutant concentration distribution along the flow with $D_0 = 0.4 \text{ km}^2/\text{day}$ and against the flow with $D_0 = 0.04 \text{ km}^2/\text{day}$ for unsteady groundwater flow with the sinusoidal form of velocity at $q=0.00001/\text{day}$ for $t > t_0$.  

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Fig. 3(a) Pollutant concentration distribution along the flow with $D_0 = 0.4 \text{ km}^2/\text{day}$ and against the flow with $D_0 = 0.04 \text{ km}^2/\text{day}$ for unsteady groundwater flow with the exponentially decreasing form of velocity at $q=0.0001/\text{day}$ for $0<t \leq t_0$.

Fig. 3(b) Pollutant concentration distribution along the flow with $D_0 = 0.4 \text{ km}^2/\text{day}$ and against the flow with $D_0 = 0.04 \text{ km}^2/\text{day}$ for unsteady groundwater flow the exponentially decreasing form of velocity at $q=0.00001/\text{day}$ for $t>t_0$.

4. Conclusions
A one-dimensional advective-dispersive solute transport model is solved for temporally dependent non-reactive source concentration. The source is considered as a pulse type input concentration in the intermediate portion of the aquifer. Initially the aquifer is not solute free. The
contaminant concentration is predicted along and against the flow in the domains $x \geq x_0$ and $0 \leq x \leq x_0$. The governing equation is solved analytically, using the Laplace Transform Technique (LTT). Temporally dependent forms of velocities, such as sinusoidal as well as exponential decreasing forms of velocities, are considered in which the sinusoidal form represents the seasonal variation in a year in tropical regions. We compare the solute concentration distribution pattern for both types of velocities. The analytical solution may be used as a preliminary predictive tool in groundwater management.

For the sinusoidal form of velocity, it is found that the contaminant concentration decreases with time and distance traveled in the presence of source contaminants. At the specified location, temporally dependent source concentration decreases as time increases and emerges at the common point near the source. The solute concentration goes on decreasing in the domain beyond this location. It is also observed that the trend is almost similar in the domain bounded by this location. Once the source concentration is eliminated, the contaminant concentration increases initially and attains the maximum level and then starts to decrease and goes on decreasing and tends towards minimum and harmless concentration. For the exponentially decreasing form of velocity, it is observed that the contaminant concentration follows almost the same trends in the presence and absence of source contaminants, respectively, in both the domains. It is observed the rate of decreasing tendency for the sinusoidal form of velocity is slightly slower than for the exponential form of velocity in both the domains.

References

